Exercise 6

Find the linear approximation of the function $g(x) = \sqrt[3]{1+x}$ at $a = 0$ and use it to approximate the numbers $\sqrt[3]{0.95}$ and $\sqrt[3]{1.1}$. Illustrate by graphing g and the tangent line.

Solution

Start by finding the corresponding y-value to $x = 0$.

$$g(0) = \sqrt[3]{1+0} = 1$$

Then find the slope of the tangent line to the function at $x = 0$ by computing $g'(x)$,

$$\begin{aligned} g'(x) &= \frac{d}{dx}\sqrt[3]{1+x} \\ &= \frac{d}{dx}(1+x)^{1/3} \\ &= \frac{1}{3}(1+x)^{-2/3} \cdot \frac{d}{dx}(1+x) \\ &= \frac{1}{3(1+x)^{2/3}} \cdot (1) \\ &= \frac{1}{3(1+x)^{2/3}}, \end{aligned}$$

and plugging in $x = 0$.

$$g'(0) = \frac{1}{3(1+0)^{2/3}} = \frac{1}{3}$$

Now use the point-slope formula to obtain the equation of the line going through $(0, 1)$ with slope $1/3$.

$$\begin{aligned} y - g(0) &= g'(0)(x-0) \\ y - 1 &= \frac{1}{3}x \\ y &= \frac{1}{3}x + 1 \end{aligned}$$

Therefore, the linearization of the function $g(x)$ at $a = 0$ is

$$L(x) = \frac{1}{3}x + 1.$$

Compare the function and its linearization for $\sqrt[3]{0.95} = \sqrt[3]{1+(-0.05)}$.

$$f(-0.05) = \sqrt[3]{0.95} \approx 0.983048 \qquad L(-0.05) = \frac{1}{3}(-0.05) + 1 \approx 0.983333$$

Compare the function and its linearization for $\sqrt{1.1} = \sqrt{1+0.1}$.

$$f(0.1) = \sqrt[3]{1.1} \approx 1.03228 \qquad L(0.1) = \frac{1}{3}(0.1) + 1 = 1.03333$$

Below is a plot of the function and the linearization at $a = 0$ versus x.